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Hyperbolic Closed Characteristics on Compact Convex Smooth Hypersurfaces in $\mathbf{R}^{2n}$

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In this paper we prove that on every compact C^2 hypersurface in $\mathbf{R}^{2n}$ bounding a convex set with non-empty interior, either there exists a sequence of variationally visible hyperbolic closed characteristcs with their minimal periods tending to

1. MAIN RESULTS

In this paper, we study the hyperbolic and nonhyperbolic closed characteristics on any given compact C^2 hypersurfaces in $\mathbf{R}^{2n}$ with $n \geqslant 2$ bounding a convex set with nonempty interior.

For any function $H \in C^1(\mathbf{R}^{2n}, \mathbf{R}) \cap C^2(\mathbf{R}^{2n} \backslash \{0\}, \mathbf{R})$, let (τ, x) be a non-constant solution of the following periodic problem of the Hamiltonian system with $\tau > 0$ and absolutely continuous x:

$$\begin{cases} \dot{x}(t) = JH'(x(t)), & \forall t \in \mathbf{R}, \\ x(\tau) = x(0). \end{cases} \tag{1.1}$$

Here $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ is the standard symplectic matrix on $\mathbf{R}^{2n}$ with I_n being the identity matrix on $\mathbf{R}^n$. Suppose the orbit $\mathcal{O}_x = \{x(t) \in \mathbf{R}^{2n} \mid t \in \mathbf{R}\}$ of x does not pass through the origin. Denote by $\gamma_x(t)$ the fundamental solution of the linearized system of (1.1) with $\gamma_x(0) = I_{2n}$,

$$\dot{y} = JB(t)y, \qquad \text{for} \quad y \in \mathbf{R}^{2n}, \tag{1.2}$$

where $B(t) = H''(x(t))$ for all $t \in \mathbf{R}$. Then γ_x is a C^1 path in the symplectic group $\mathrm{Sp}(2n)$. We call respectively the path γ_x and the matrix $\gamma_x(\tau)$ the

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associated symplectic path and the *Poincaré matrix* of the solution (τ, x) of (1.1). The eigenvalues of $\gamma_x(\tau)$ are called the Floquet multipliers of (τ, x) as usual. Note that since (1.1) is autonomous and x is non-constant, 1 must be a Floquet mutiplier of (τ, x).

DEFINITION 1.1. A non-constant solution (τ, x) of the problem (1.1) is *hyperbolic* if 1 is a double Floquet multiplier and all the other Floquet multipliers of (τ, x) are not on the unit circle $\mathbf{U}$ in the complex plane $\mathbf{C}$.

Now let Σ be a C^2 compact hypersurface in $\mathbf{R}^{2n}$ bounding a convex set C with non-empty interior. We denote the set of all such hypersurfaces in $\mathbf{R}^{2n}$ by $\mathscr{H}(2n)$. For $x \in \Sigma$ let $N_\Sigma(x)$ be the unit vector on the outward normal to Σ at x. We consider the given energy problem of finding $\tau > 0$ and an absolutely continuous curve $x: [0, \tau] \to \mathbf{R}^{2n}$ such that

$$\begin{cases} \dot{x}(t) = JN_\Sigma(x(t)), & x(t) \in \Sigma, \quad \forall t \in \mathbf{R}, \\ x(\tau) = x(0). \end{cases} \tag{1.3}$$

DEFINITION 1.2. A solution (τ, x) of the problem (1.3) with τ being the minimal period of x is called a closed characteristic on Σ. Denote by $\mathscr{J}(\Sigma)$ the set of all closed characteristics on Σ.

To cast the problem (1.3) into a Hamiltonian version, we follow Chapter V of [Ek3]. For a given $\Sigma \in \mathscr{H}(2n)$ bounding a convex set C, without loss of generality we assume the origin is in the interior of C. Let $j_C: \mathbf{R}^{2n} \to [0, +\infty)$ be the gauge function of C defined by

$$j_C(0) = 0 \qquad \text{and} \qquad j_C(x) = \inf\left\{\lambda \,\middle|\, \frac{x}{\lambda} \in C\right\} \qquad \text{for} \quad x \neq 0.$$

Fix a constant α satisfying $1 < \alpha < 2$ in this paper. As usual we define the Hamiltonian function $H_\alpha: \mathbf{R}^{2n} \to [0, +\infty)$ by

$$H_\alpha(x) = j_C(x)^\alpha, \qquad \forall x \in \mathbf{R}^{2n}.$$

Then $H_\alpha \in C^1(\mathbf{R}^{2n}, \mathbf{R}) \cap C^2(\mathbf{R}^{2n}\backslash\{0\}, \mathbf{R})$ is convex and $\Sigma = H_\alpha^{-1}(1)$. It is well known that the problem (1.3) is equivalent to the following problem

$$\begin{cases} \dot{x}(t) = JH'_\alpha(x(t)), & H_\alpha(x(t)) = 1, \quad \forall t \in \mathbf{R}, \\ x(\tau) = x(0). \end{cases} \tag{1.4}$$

Denote by $\mathscr{J}(\Sigma, \alpha)$ the set of all solutions (τ, x) of the problem (1.4) with τ being the minimal period of x. Note that elements in $\mathscr{J}(\Sigma)$ and $\mathscr{J}(\Sigma, \alpha)$ are one to one correspondent to each other. By Proposition I.6.13 of [Ek3], the Floquet multipliers with their multiplicity and Krein signs of

$(\tau, x) \in \mathscr{J}(\Sigma)$ do not depend on the particular choice of the Hamiltonian function in (1.4). Thus the hyperbolicity and ellipticity of elements in $\mathscr{J}(\Sigma)$ can be determined through corresponding elements in $\mathscr{J}(\Sigma, \alpha)$.

For any $(\tau, x) \in \mathscr{J}(\Sigma, \alpha)$ and $m \in \mathbf{N} = \{1, 2, ...\}$, the mth iteration x^m of x is defined by

$$x^m(t) = x(t - j\tau), \qquad \text{for} \quad j\tau \leqslant t \leqslant (j+1)\tau, \qquad 0 \leqslant j \leqslant m-1. \quad (1.5)$$

This is simply x itself viewed as an $m\tau$-periodic function. An effective way to distinguish different elements in $\mathscr{J}(\Sigma, \alpha)$ is to compare the Maslov-type index sequences of their iterations. For periodic solutions of any periodic Hamiltonian system, this index theory was defined by C. Conley, E. Zehnder, and the author in [CZ], [LZ], and [Lo1] (cf. also [Lo4–7]). This index theory assigns to the iteration sequence $\{x^m\}$ of each solution $(\tau, x) \in \mathscr{J}(\Sigma, \alpha)$ a sequence of pairs of integers $\{(i_{m\tau}(x^m), \nu_{m\tau}(x^m))\}_{m \in \mathbf{N}}$ through the associated symplectic path γ_x of x. We give a brief introduction of this index theory in Section 5.

To solve the given energy problem (1.4), we consider the following fixed period problem

$$\begin{cases} \dot{z}(t) = JH'_\alpha(z(t)), & \forall t \in \mathbf{R}, \\ z(1) = z(0). \end{cases} \quad (1.6)$$

Define the dual function H^*_α of H_α by $H^*_\alpha(x) = \sup_{y \in \mathbf{R}^{2n}} \{\langle y, x\rangle - H(y)\}$, where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbf{R}^{2n}$. For $1 < \alpha < 2$, define $E_\alpha = \{u \in L^{(\alpha-1)/\alpha}(\mathbf{R}/\mathbf{Z}), \mathbf{R}^{2n}) \mid \int_0^1 u\, dt = 0\}$. Define the Clarke–Ekeland dual action functional $f_\alpha : E_\alpha \to \mathbf{R}$ by

$$f_\alpha(u) = \int_0^1 \{\tfrac{1}{2}\langle Ju, \Pi u\rangle + H^*_\alpha(-Ju)\}\, dt, \quad (1.7)$$

where Πu is defined by $d/dt\, \Pi u = u$ and $\int_0^1 \Pi u\, dt = 0$. Then $f_\alpha \in C^2(E_\alpha, \mathbf{R})$. Suppose $u \in E_\alpha$ is a nontrivial critical point of f_α. By [Ek3], there exists $\xi_u \in \mathbf{R}^{2n}$ such that $z_u(t) = \Pi u(t) + \xi_u$ is a 1-periodic solution of the problem (1.6). Denote the Ekeland index and nullity (cf. [Ek1, 2, 3] for details) of f_α at u by $i^E_1(u)$ and $\nu^E_1(u)$ respectively. We denote the corresponding Maslov-type indices of z_u by $(i_1(z_u), \nu_1(z_u))$. Let $h = H_\alpha(z_u(t))$ and $1/m$ be the minimal period of z_u for some $m \in \mathbf{N}$. Define

$$x_u(t) = h^{-1/\alpha} z_u(h^{(2-\alpha)/\alpha} t) \qquad \text{and} \qquad \tau = \frac{1}{m} h^{(\alpha-2)/\alpha}. \quad (1.8)$$

Then there hold $x_u(t) \in \Sigma$ for all $t \in \mathbf{R}$ and $(\tau, x_u) \in \mathscr{J}(\Sigma, \alpha)$. Note that the period 1 of z_u corresponds to the period $m\tau$ of the solution $(m\tau, x_u^m)$ of (1.4) with minimal period τ. The following Lemmata will be proved in the Section 3.

LEMMA 1.3. *For u and $z = z_u$ defined above, there hold*

$$i_1(z) = i_1^E(u) + n \qquad \text{and} \qquad \nu_1(z) = \nu_1^E(u). \tag{1.9}$$

LEMMA 1.4. *For $z = z_u$, $x = x_u$, τ and m defined above, there hold*

$$i_{m\tau}(x^m) = i_1(z) \qquad \text{and} \qquad \nu_{m\tau}(x^m) = \nu_1(z). \tag{1.10}$$

On the other hand, every solution $(\tau, x) \in \mathscr{J}(\Sigma, \alpha)$ gives rise to a sequence $\{z_m^x\}_{m \in \mathbf{N}}$ of solutions of the problem (1.6), and a sequence $\{u_m^x\}_{m \in \mathbf{N}}$ of critical points of f_α defined by

$$z_m^x(t) = (m\tau)^{-1/(2-\alpha)}\, x(m\tau t), \tag{1.11}$$

$$u_m^x(t) = (m\tau)^{(\alpha-1)/(2-\alpha)}\, \dot{x}(m\tau t). \tag{1.12}$$

Applying the above argument on τ-periodic function spaces, from the corresponding Lemma 1.3 we obtain

$$n \leqslant i_\tau(x), \qquad \forall (\tau, x) \in \mathscr{J}(\Sigma, \alpha). \tag{1.13}$$

For a direct proof of (1.13) we refer to [DL].

Following Section V.3 of [Ek3], denote by "ind" the S^1-action cohomology index theory for S^1-invariant subset of E_α defined in [Ek3] (cf. also [FR] for the original definition). For $[f_\alpha]_c \equiv \{u \in E_\alpha \mid f_\alpha(u) \leqslant c\}$ define

$$c_k = \inf\{c < 0 \mid \operatorname{ind}([f_\alpha]_c) \geqslant k\}. \tag{1.14}$$

Then there hold

$$-\infty < \min_{u \in E_\alpha} f_\alpha(u) = c_1 \leqslant c_2 \leqslant \cdots \leqslant c_k \leqslant c_{k+1} \leqslant \cdots < 0, \tag{1.15}$$

$$c_k \to 0 \qquad \text{as} \quad k \to +\infty, \tag{1.16}$$

and all the c_k's are critical values of f_α. By Theorem V.3.4 of [Ek3], for each $k \in \mathbf{N}$ there exists a function $u_k \in E_\alpha$ such that there hold

$$f'_\alpha(u_k) = 0 \qquad \text{and} \qquad f_\alpha(u_k) = c_k, \tag{1.17}$$

$$i_1^E(u_k) \leqslant 2k - 2 \leqslant i_1^E(u_k) + \nu_1^E(u_k) - 1, \tag{1.18}$$

Definition 1.5. $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$ is *(m, k)-variationally visible*, if there exist some m and $k \in \mathbf{N}$ such that for u_m^x defined by (1.12) there hold

$$f'_\alpha(u_m^x) = 0 \qquad \text{and} \qquad f_\alpha(u_m^x) = c_k, \tag{1.19}$$

$$i_{m\tau}(x^m) \leqslant 2k - 2 + n \leqslant i_{m\tau}(x^m) + \nu_{m\tau}(x^m) - 1. \tag{1.20}$$

We denote by $\mathscr{V}_{m,k}(\Sigma, \alpha)$ the set of all (m, k)-variationally visible closed characteristics on Σ, and define $\mathscr{V}(\Sigma, \alpha) = \bigcup_{m,k \in \mathbf{N}} \mathscr{V}_{m,k}(\Sigma, \alpha)$.

Remark 1.6. Note that any $(\tau, x) \in \mathscr{V}_{m,k}(\Sigma, \alpha)$ is k-essential in the sense of Definition V.3.6 of [Ek3]. But variational visibility yields more precise information needed in our study.

Next we introduce two new global invariants of Σ with parameter $\alpha > 1$.

Definition 1.7. For $\Sigma \in \mathscr{H}(2n)$ and $\alpha > 1$, define the *variationally visible index cover set* $\mathscr{I}(\Sigma, \alpha)$ of (Σ, α) by

$$\mathscr{I}(\Sigma, \alpha) = \{q \in \mathbf{N} \mid \exists m, k \in \mathbf{N} \text{ and } (\tau, x) \in \mathscr{V}_{m,k}(\Sigma, \alpha) \text{ such that}$$
$$q \in [i_{m\tau}(x^m), i_{m\tau}(x^m) + \nu_{m\tau}(x^m) - 1]\}. \tag{1.21}$$

Definition 1.8. For $\Sigma \in \mathscr{H}(2n)$ and $\alpha > 1$, define the *variationally visible hyperbolic index cover set* $\mathscr{I}_h(\Sigma, \alpha)$ of (Σ, α) by

$$\mathscr{I}_h(\Sigma, \alpha) = \{q \in \mathbf{N} \mid \exists m, k \in \mathbf{N} \text{ and hyperbolic } (\tau, x) \in \mathscr{V}_{m,k}(\Sigma, \alpha) \text{ such that}$$
$$q \in [i_{m\tau}(x^m), i_{m\tau}(x^m) + \nu_{m\tau}(x^m) - 1]\}.$$

Note that the variationally visible elliptic index cover set $\mathscr{I}_e(\Sigma, \alpha)$ of (Σ, α) can be defined similarly. By (1.18) and Definition 1.5 there always holds

$$(2\mathbf{N} - 2 + n) \cup \mathscr{I}_h(\Sigma, \alpha) \cup \mathscr{I}_e(\Sigma, \alpha) \subset \mathscr{I}(\Sigma, \alpha) \subset \mathbf{N}. \tag{1.22}$$

For $1 < \alpha < 2$, by Proposition 4.1 below, there holds $\nu_{m\tau}(x^m) = 1$ for all hyperbolic $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)\}$. Thus together with Lemmata 1.3 and 1.4 we obtain

$$\mathscr{I}_h(\Sigma, \alpha) = \{i_{m\tau}(x^m) \in \mathbf{N} \mid \exists m, k \in \mathbf{N} \text{ and hyperbolic } (\tau, x) \in \mathscr{V}_{m,k}(\Sigma, \alpha)\}. \tag{1.23}$$

Fix $1 < \alpha < 2$. Define $\Sigma = \{x = (x_1, ..., x_n) \in \mathbf{R}^{2n} \mid \frac{1}{2} \sum_{k=1}^n \omega_i |x_i|^2 = 1\}$ with $\omega_k > 0$ for $1 \leqslant k \leqslant n$.

If ω_i and ω_j are rationally independent whenever $i \neq j$, Σ is a weakly nonresonant ellipsoid. Then there are precisely n closed characteristics on

Σ, all of them are elliptic, and there hold $\mathscr{I}_h(\Sigma, \alpha) = \emptyset$ and $\mathscr{I}(\Sigma, \alpha) = \mathscr{I}_e(\Sigma, \alpha) = 2\mathbf{N} - 2 + n$.

If $\omega_k = \omega_1$ for all k, Σ is a sphere. Then passing through every point on the sphere Σ there is a closed characteristic. All of these characteristics are elliptic, and possess the same minimal period. There hold $\mathscr{I}_h(\Sigma, \alpha) = \emptyset$ and $\mathscr{I}(\Sigma, \alpha) = \mathscr{I}_e(\Sigma, \alpha) = \mathbf{N} - 1 + n$.

The following is the main result of this paper.

THEOREM 1.9. *For any $\Sigma \in \mathscr{H}(2n)$ with $n \geq 2$ and $1 < \alpha < 2$, suppose the minimal periods of all hyperbolic elements in $\mathscr{V}(\Sigma, \alpha)$ are uniformally bounded from above. Then there holds*

$$\mathscr{I}(\Sigma, \alpha) \backslash \mathscr{I}_h(\Sigma, \alpha) \neq \emptyset. \tag{1.24}$$

A direct consequence of this theorem is:

COROLLARY 1.10. *On every C^2 compact hypersurface Σ in $\mathbf{R}^{2n}$ with $n \geq 2$ bounding a convex set with non-empty interior, either there exists a sequence of variationally visible hyperbolic closed characteristics with their minimal periods tending to infinity, or there exists at least one variationally visible nonhyperbolic closed characteristic.*

The existence of at least one closed characteristic on any $\Sigma \in \mathscr{H}(2n)$ was first established by P. Rabinowitz [Ra] (for star-shaped hypersurfaces) and A. Weinstein [We] independently in 1978. The basic result of Corollary 1.10 was first proved by I. Ekeland as Theorem IV.1 of [Ek2]. But with all due respect, there is a flaw in the iteration formulae given in Corollary IV.8 of [Ek1] and (IV.17) of [Ek2] (cf. Remark 4.3 for more details and comments). This paper gives a new derivation (and hopefully correct) of this iteration formula in a more general setting, whereby the result is put on safer ground. For other related results on the existence of at lest one elliptic (i.e., all the Floquet multipliers are on the unit circle) closed characteristic on some restricted subclasses of $\mathscr{H}(2n)$, we refer to [Ek2], [Ek3], [De], and [DDE]. These results are related to the classical Problem 3 mentioned at the end of the celebrated book [Ek3]. Note that in [Vi1], interesting Morse index study has been carried out for Hamiltonian orbits on star-shaped hypersurfaces when all these orbits are non-degenerate. Note that our iteration formula Theorem 2.1 works for all Hamiltonian systems and has no such non-degeneracy conditions.

To prove Theorem 1.9 we first establish precise iteration formulae of Maslov-type indices for any hyperbolic closed characteristics on Σ (Theorem 2.1 and Proposition 4.1). Here note that these iteration formulae hold for any energy hypersurfaces which need not be convex. By Lemmata 1.3

and 1.4, we show that under the conditions of Theorem 1.9, the sequences of Maslov-type indices for iterations of all such hyperbolic $(\tau, x)\in\mathscr{V}(\Sigma, \alpha)$ must fall into the following at most finitely many different patterns:

$$i_{m\tau}(x^m)=m(q+1)-1, \qquad \nu_{m\tau}(x^m)=1, \quad \forall m\in\mathbf{N}, \quad n\leqslant q\leqslant q_0, \qquad (1.25)$$

for some $q_0\in\mathbf{N}$ depending only on Σ and α. By this pattern finiteness result, there exists a subset $\mathscr{B}(\Sigma, \alpha)$ of $\mathbf{N}$ depending also on the parity of n consists of certain integers appeared in (1.25) such that there holds $\mathscr{I}_h(\Sigma, \alpha)\subset\mathscr{B}(\Sigma, \alpha)$ (Proposition 4.4). Now if (1.24) does not hold, by (1.22) the set $2\mathbf{N}-2+n$ would become a subset of $\mathscr{B}(\Sigma, \alpha)$. By the properties of these two sets, we prove that this contradicts to the infinitude of prime numbers. Our proof uses also some idea and results of I. Ekeland in [Ek2, 3]. For the reader's convenience, in the Section 5, the appendix of this paper, we give a brief review of the Maslov-type index theory and its basic properties used in this paper.

2. MASLOV-TYPE INDICES FOR ITERATIONS OF HYPERBOLIC PERIODIC SOLUTIONS

For $H\in C^1(\mathbf{R}^{2n}, \mathbf{R})$, fix a non-constant hyperbolic solution (τ, x) of the autonomous Hamiltonian system (1.1). Denote by γ_x the fundamental solution of the linear system (1.2) with $B(t)=H''(x(t))$. Suppose H is C^2 near $\mathcal{O}_x$. In this Section we study the iteration property of Maslov-type indices of (τ, x) through its associated symplectic path γ_x. Without loss of generality, in this Section we suppose $\tau=1$.

For any path $\gamma\in C([0, 1], \mathrm{Sp}(2n))$ we define its iteration $\tilde{\gamma}\in C([0, +\infty), \mathrm{Sp}(2n))$ by

$$\tilde{\gamma}(t)=\gamma(t-j)\,\gamma(1)^j, \qquad \text{for} \quad j\leqslant t\leqslant(j+1), \quad 0\leqslant j\in\mathbf{Z}. \qquad (2.1)$$

Note that by the uniqueness theorem of the initial value problem of ordinary differential equations, the symplectic path γ_x associated to the solution x satisfies $(\gamma_x|_{[0,1]})^{\sim}=\gamma_x$.

By Definition 1.1 of the hyperbolicity, all the eigenvalues of the symplectic matrix $\gamma_x(1)$ are away from the unit circle $\mathbf{U}$ except that 1 is a double eigenvalue of $\gamma_x(1)$. Thus there exist matrices $P\in\mathrm{Sp}(2n)$ and $M\in\mathrm{Sp}(2n-2)$ such that

$$\gamma_x(1)=P(\hat{N}_2\diamond M)\,P^{-1}, \qquad (2.2)$$

where the definition of the $\diamond$-product is given in Section 5, $\sigma(M)\cap\mathbf{U}=\emptyset$, $\hat{N}_2=N_2(a)$ with $a=0$, 1, or -1, and $N_2(a)$ is defined by

$$N_2(a)=\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}, \qquad \forall a\in\mathbf{R}. \tag{2.3}$$

Since Sp$(2n)$ is path connected, there is a path $P_1\in\mathscr{P}_1(2n)$ (cf. Section 5) such that $P_1(1)=P$. Define

$$g_1(s)=P_1(s)^{-1}\,\gamma_x(1)\,P_1(s), \qquad \forall s\in[0,1].$$

Then g_1 connects $\gamma_x(1)$ to $\hat{N}_2\diamond M$ in Sp$(2n)$, and $g_1(s)$ possesses precisely the same spectrum as that of $\gamma_x(1)$ for every $s\in[0,1]$.

Since M possesses no eigenvalue on the unit circle, by a small perturbation, by a sufficiently short path $g_2\colon[0,1]\to\mathrm{Sp}(2n-2)$ we can connect $M=g_2(0)$ to a matrix $g_2(1)\in\mathrm{Sp}(2n-2)$ such that $g_2(1)$ possesses only simple eigenvalues and $\sigma(g_2(s))\cap\mathbf{U}=\emptyset$ for all $s\in[0,1]$. Then we get a path $P_2\in\mathscr{P}_1(2n-2)$ such that for some integer $q\in[n/2,n]$ there holds

$$g_2(1)=P_2(1)(Q_1\diamond\cdots\diamond Q_q)\,P_2(1)^{-1},$$

where $Q_i=D(\rho_i)$ or $Q_i=\mathrm{diag}(\rho_iR(\theta_i),\rho_i^{-1}R(\theta_i))$ with $\rho_i\in\mathbf{R}\backslash\{0,\pm1\}$ and $\theta_i\in(0,\pi)\cup(\pi,2\pi)$ for each $i\in[1,q]$. Here $R(\theta)$ and $D(a)$ for $\theta\in\mathbf{R}$ and $a\in\mathbf{R}\backslash\{0\}$ are defined in Section 5. Define

$$g_3(s)=P_2(s)^{-1}\,g_2(1)\,P_2(s), \qquad \forall s\in[0,1].$$

Then g_3 connects $g_3(0)=g_2(1)$ to $g_3(1)=Q_1\diamond\cdots\diamond Q_q$ in Sp$(2n-2)$, and $\sigma(g_3(s))\cap\mathbf{U}=\emptyset$ for all $s\in[0,1]$.

If $Q_i=\mathrm{diag}(\rho_iR(\theta_i),\rho_i^{-1}R(\theta_i))$ with $\rho_i\in\mathbf{R}\backslash\{0,\pm1\}$ and $\theta_i\in(0,\pi)\cup(\pi,2\pi)$, define

$$h_i(s)=Q_i[\mathrm{diag}(R(-s\theta_i),R(-s\theta_i))], \qquad \forall t\in[0,1].$$

Then h_i connects Q_i to $D(\rho_i)\diamond D(\rho_i)$ with $\sigma(h_i(s))\cap\mathbf{U}=\emptyset$ for all $s\in[0,1]$.

By the topological structure of Sp$(2)^*$ via the $\mathbf{R}^3$-cylinderical coordinate representation introduced in [Lo2], there is a path $u_i\colon[0,1]\to\mathrm{Sp}(2)$ connecting $u_i(0)=D(\rho_i)$ to $u_i(1)=D(\varepsilon_i2)$, with $\sigma(u_i(s))\cap\mathbf{U}=\emptyset$ for all $s\in[0,1]$, where $\varepsilon_i=\pm1$ if $\pm\rho_i>0$.

In such a way, we have proved that there is a path $g_4\colon[0,1]\to\mathrm{Sp}(2n-2)$ connecting $g_4(0)=g_3(1)=Q_1\diamond\cdots\diamond Q_q$ to a $\diamond$-product $g_4(1)$ of $D(2)$'s and $D(-2)$'s in Sp$(2n-2)$ with $\sigma(g_4(s))\cap\mathbf{U}=\emptyset$ for all $s\in[0,1]$.

Whenever there are two $D(-2)$'s appearing in $g_4(1)$, by a rotation path similar to that defined in the definition of the path h_i, we can connects

$g_4(1)$ to a new matrix so that these two $D(-2)$'s have been changed to two $D(2)$'s and this connecting path possesses no eigenvalue on the unit circle at any time. Thus by induction, we obtain a path $g_5\colon [0,1]\to \mathrm{Sp}(2n-2)$ which connects $g_5(0)=g_4(1)$ to one of the following two matrices in $\mathrm{Sp}(2n-2)$:

$$D(2)^{\diamond(n-2)}\diamond D(-2), \qquad D(2)^{\diamond(n-1)},$$

and there holds $\sigma(g_5(s))\cap \mathbf{U}=\varnothing$ for all $s\in[0,1]$.

Define

$$g(s)=[\hat{N}_2\diamond(g_5 * g_4 * g_3 * g_2)] * g_1(s), \qquad \forall\ s\in[0,1]. \tag{2.4}$$

The definition of the joint path is given in Section 5. Then $g\in C([0,1],\mathrm{Sp}(2n))$ connects $g(0)=\gamma_x(1)$ to one of the following two matrices in $\mathrm{Sp}(2n)$:

$$\hat{N}_2\diamond D(2)^{\diamond(n-2)}\diamond D(-2), \tag{2.5}$$

or

$$\hat{N}_2\diamond D(2)^{\diamond(n-1)}, \tag{2.6}$$

and for each $s\in[0,1]$ there exist $F(s)\in\mathrm{Sp}(2n)$ and $G(s)\in\mathrm{Sp}(2n-2)$ such that $g(s)=F(s)[\hat{N}_2\diamond G(s)]\,F(s)^{-1}$ and $\sigma(G(s))\cap\mathbf{U}=\varnothing$.

Define paths in $\mathrm{Sp}(2)$ by

$$\phi_{\pm}(t)=N_2(\mp t), \qquad \psi_{+}(t)=D(1+t), \qquad \text{for} \quad 0\leqslant t\leqslant 1, \tag{2.7}$$

$$\zeta_a(t)=R(2a\pi t), \qquad \text{for} \quad 0\leqslant t\leqslant 1 \qquad \text{and} \qquad a\in\mathbf{R}, \tag{2.8}$$

$$\psi_{-}(t)=\hat{\psi}_{-} * \zeta_{1/2}(t), \qquad \hat{\psi}_{-}(t)=D(-(1+t)), \qquad \text{for} \quad 0\leqslant t\leqslant 1. \tag{2.9}$$

We continue our study in three cases:

Case 1. $\hat{N}_2=N_2(1)$.

Subcase 1.1. $g(1)$ is given by (2.5).

For $k\in\mathbf{Z}$ define a new path $\beta\colon[0,1]\to\mathrm{Sp}(2n)$ by

$$\beta(t)=([\phi_{-} * \zeta_k]\diamond[\psi_{+}]^{\diamond(n-2)}\diamond\psi_{-})(t). \tag{2.10}$$

By definition we obtain

$$i_1(\phi_{-} * \zeta_k)=2k-1, \qquad i_1(\psi_{+})=0, \qquad i_1(\psi_{-})=1,$$

$$\nu_1(\phi_{-} * \zeta_k)=1, \qquad \nu_1(\psi_{+})=0, \qquad \nu_1(\psi_{-})=0.$$

Adding them up yields

$$i_1(\beta)=2k, \qquad \nu_1(\beta)=1.$$

Denote by the same name the iterations of these paths defined by (2.1). Then for any $m\in\mathbf{N}$ we obtain

$$\begin{cases} i_m(\phi_- * \zeta_k)=2km-1, & i_m(\psi_+)=0, \quad i_m(\psi_-)=m, \\ \nu_m(\phi_- * \zeta_k)=1, & \nu_m(\psi_+)=0, \quad \nu_m(\psi_-)=0. \end{cases} \tag{2.12}$$

Thus there hold

$$i_m(\beta)=m(2k+1)-1, \qquad \nu_m(\beta)=1, \qquad \forall m\in\mathbf{N}. \tag{2.13}$$

By the definition (2.4) of the path g and the fact $(\mathrm{Sp}(2)^0_-)^m\subset\mathrm{Sp}(2)^0_-$ for all $m\in\mathbf{N}$ there hold

$$g(0)=\gamma_x(1), \qquad g(1)=\beta(1), \tag{2.14}$$

$$\dim\ker(g(s)^m-I_{2n})=1, \qquad \forall s\in[0,1], \quad m\in\mathbf{N}. \tag{2.15}$$

Thus in this subcase there holds (cf. Definition 5.3)

$$\gamma_x(1)\simeq N_2(1)\diamond D(2)^{\diamond(n-2)}\diamond D(-2). \tag{2.16}$$

Therefore (2.11) yields

$$i_1(\gamma_x)\in 2\mathbf{Z}. \tag{2.17}$$

Define $k=i_1(\gamma_x)/2$. Then from (2.11), we obtain $i_1(\beta)=i_1(\gamma_x)=2k$. Together with (2.14), (2.15), (2.16), and Proposition 5.9, we obtain $\beta\sim\gamma_x$ on $[0,1]$ along g. Now this homotopy extends to $[0,1]\times[0,m]$ for all $m\in\mathbf{N}$ via the definition (2.1). Thus by (2.13), (2.15), and Proposition 5.10 we obtain

$$i_m(\gamma_x)=i_m(\beta)=m(2k+1)-1=m(i_1(\gamma_x)+1)-1, \qquad \nu_m(\gamma_x)=1, \quad \forall m\in\mathbf{N}. \tag{2.18}$$

Subcase 1.2. $g(1)$ is given by (2.6).

In this subcase there holds

$$\gamma_x(1)\simeq N_2(1)\diamond D(2)^{\diamond(n-1)}. \tag{2.19}$$

Then $i_1(\gamma_x)$ must be odd. In the discussion of the Subcase 1.1, we define k by $2k-1=i_1(\gamma_x)$ and $\beta(t)=([\phi_- * \zeta_k]\diamond[\psi_+]^{\diamond(n-1)})(t)$ for all $t\in[0,1]$. Similarly we obtain

$$i_m(\gamma_x)=2km-1=m(i_1(\gamma_x)+1)-1, \qquad \nu_m(\gamma_x)=1, \qquad \forall m\in\mathbf{N}. \quad (2.20)$$

Case 2. $\hat{N}_2=I_2$.

Subcase 2.1. $g(1)$ is given by (2.5).

In this subcase there holds

$$\gamma_x(1)\simeq I_2\diamond D(2)^{\diamond(n-2)}\diamond D(-2). \quad (2.21)$$

So $i_1(\gamma_x)$ must be even. In the discussion of Subcase 1.1, we define $k=i_1(\gamma_x)/2$ and $\beta(t)=(\zeta_k\diamond[\psi_+]^{\diamond(n-2)}\diamond\psi_-)(t)$ for all $t\in[0,1]$. From the discussion in the Subcase 1.1 and the facts $i_m(\zeta_k)=2km-1$ and $\nu_m(\phi_- * \zeta_k)=1$, we obtain $i_m(\beta)=m(2k+1)-1$ and $\nu_m(\beta)=2$ for all $m\in\mathbf{N}$. Thus similarly there hold

$$i_m(\gamma_x)=m(2k+1)-1=m(i_1(\gamma_x)+1)-1, \qquad \nu_m(\gamma_x)=2, \quad \forall m\in\mathbf{N}. \quad (2.22)$$

Subcase 2.2. $g(1)$ is given by (2.6).

In this subcase there holds

$$\gamma_x(1)\simeq I_2\diamond D(2)^{\diamond(n-1)}. \quad (2.23)$$

Thus $i_1(\gamma_x)$ must be odd. In the discussion of Subcase 1.1, we define k by $2k-1=i_1(\gamma_x)$ and $\beta(t)=(\zeta_k\diamond[\psi_+]^{\diamond(n-1)})(t)$ for all $t\in[0,1]$. Then similarly we obtain

$$i_m(\gamma_x)=2km-1=m(i_1(\gamma_x)+1)-1, \qquad \nu_m(\gamma_x)=2, \qquad \forall m\in\mathbf{N}. \quad (2.24)$$

Case 3. $\hat{N}_2=N_2(-1)$.

Subcase 3.1. $g(1)$ is given by (2.5).

In this subcase there holds

$$\gamma_x(1)\simeq N_2(-1)\diamond D(2)^{\diamond(n-2)}\diamond D(-2). \quad (2.25)$$

Then $i_1(\gamma_x)$ must be odd. In the discussion of the subcase 1.1, we define k by $2k+1=i_1(\gamma_x)$ and $\beta(t)=([\phi_+ * \zeta_k]\diamond[\psi_+]^{\diamond(n-2)}\diamond\psi_-)(t)$ for all

$t \in [0, 1]$. Then from the facts $i_m(\phi_+ * \zeta_k) = 2km$ and $\nu_m(\phi_- * \zeta_k) = 1$, we obtain $i_m(\beta) = m(2k+1)$ and $\nu_m(\beta) = 1$ for all $m \in \mathbf{N}$. Thus similarly we obtain

$$i_m(\gamma_x) = m(2k+1) = m\, i_1(\gamma_x), \qquad \nu_m(\gamma_x) = 1, \qquad \forall m \in \mathbf{N}. \tag{2.26}$$

Subcase 3.2. $g(1)$ is given by (2.6).

In this subcase there holds

$$\gamma_x(1) \simeq N_2(-1) \diamond D(2)^{\diamond(n-1)}. \tag{2.27}$$

So $i_1(\gamma_x)$ must be even. In the discussion of Subcase 1.1, we define $k = i_1(\gamma_x)/2$ and $\beta(t) = ([\phi_+ * \zeta_k] \diamond [\psi_+]^{\diamond(n-1)})(t)$. Then similarly we obtain

$$i_m(\gamma_x) = 2km = m i_1(\gamma_x), \qquad \nu_m(\gamma_x) = 1, \qquad \forall m \in \mathbf{N}. \tag{2.28}$$

Summarizing these discussions we have proved the following

THEOREM 2.1. *Let (τ, x) be a non-constant hyperbolic solution of the autonomous Hamiltonian system* (1.1) *with H being C^2 near $\mathcal{O}_x$. Then for all $m \in \mathbf{N}$ the iterated Maslov-type index sequence $\{(i_{m\tau}(x^m), \nu_{m\tau}(x^m))\}_{m\in\mathbf{N}}$ satisfies*

$$i_{m\tau}(x^m) = m(i_\tau(x)+1) - 1, \quad \nu_{m\tau}(x^m) = 1, \quad \textit{if} \quad \gamma_x(\tau) \simeq N_2(1) \diamond M, \tag{29}$$

$$i_{m\tau}(x^m) = m(i_\tau(x)+1) - 1, \quad \nu_{m\tau}(x^m) = 2, \quad \textit{if} \quad \gamma_x(\tau) \simeq N_2(0) \diamond M, \tag{30}$$

$$i_{m\tau}(x^m) = m i_\tau(x), \quad \nu_{m\tau}(x^m) = 1, \quad \textit{if} \quad \gamma_x(\tau) \simeq N_2(-1) \diamond M, \tag{31}$$

where $M \in \mathrm{Sp}(2n-2)$ satisfies $\sigma(M) \cap \mathbf{U} = \varnothing$.

Proof. The conclusion follows from (2.18), (2.20), (2.22), (2.24), (2.26), or (2.28), if $\gamma_x(\tau)$ satisfies (2.16), (2.19), (2.21), (2.23), (2.25), or (2.27) respectively. ∎

Remark 2.2. Note that Theorem 2.1 does not require H to be convex.

3. BASIC PROPERTIES OF CLOSED CHARACTERISTICS

Using notations defined in the previous sections, we first prove Lemmata 1.3 and 1.4.

Proof of Lemma 1.3. The second equality of (1.9) follows from Theorem I.4.4 of [Ek3] and the Definition 5.1 of $\nu_1(z)$. We carry out the proof of the first equality of (1.9) in two cases:

Case 1. $\nu_1(z)=0$.

In this case the first equality of (1.9) was proved by V. Brousseau in [Br1] and [Br2].

Case 2. $\nu_1(z)>0$.

In this case let $\psi: [0, +\infty) \to \mathrm{Sp}(2n)$ be the associated symplectic path of z, i.e., the fundamental solution of (1.2) with $B(t)=H''_\alpha(z(t))$. Thus by definition there hold $i_1(z)=i_1(\psi)$. By Theorem 5.7 there exists a one-parameter continuous family $\psi_s \in P_1(2n)$ for $s\in[-1, 1]$ of perturbation paths of ψ such that there hold $\psi_0=\psi$, ψ_s converges to ψ in C^1 as $s\to 0$, $\psi_s(1)\in \mathrm{Sp}(2n)^*$ for $s\in[-1, 1]\backslash\{0\}$, and

$$i_1(\psi)=i_1(\psi_{-s})=i_1(\psi_s)-\nu_1(\psi), \qquad \forall s\in(0, 1]. \tag{3.1}$$

Define

$$B_s(t)=-J\dot{\psi}_s(t)\,\psi(t)^{-1}, \qquad \forall s\in[-1, 1], \quad t\in[0, 1].$$

Then $B_0(t)=B(t)$ is positive definite and continuous for all $t\in[0, 1]$, and B_s converges to B in C^0 as $s\to 0$. By reparametrizing ψ_s, without loss of generality we can further assume ψ_s to be close enough to ψ so that B_s are positive definite for all $s\in[-1, 1]$. Define

$$g_s(v)=\frac{1}{2}\int_0^1 \{\langle Jv, \pi v\rangle + \langle B_s^{-1}(t)(-Jv), -Jv\rangle\}\, dt, \qquad \forall v\in E_\alpha.$$

Then there holds $g_0=f''(u)$. Denote the Ekeland index and nullity of the functional g_s by $i_1^E(\psi_s)$ and $\nu_1^E(\psi_s)$ for $s\in[-1, 1]$ respectively. Note that by the properties of ψ_s there hold $i_1^E(u)=i_1^E(\psi)$, $\nu_1^E(u)=\nu_1^E(\psi)$, and $\nu_1(\psi_s)=\nu_1^E(\psi_s)=0$ for all $s\in[-1, 1]\backslash\{0\}$. Thus by Case 1 we obtain

$$i_1(\psi_s)=i_1^E(\psi_s)+n, \qquad \forall s\in[-1, 1]\backslash\{0\}. \tag{3.2}$$

Combining (3.2) wihth $s\in(0, 1]$ and (3.1), we obtain

$$i_1^E(\psi_s)=i_1^E(\psi_{-s})+\nu_1^E(\psi). \tag{3.3}$$

On the other hand, when $s\in(0, 1]$ is sufficiently small, there holds

$$i_1^E(\psi_s)-\nu_1^E(\psi)\leqslant i_1^E(\psi)\leqslant i_1^E(\psi_{-s}).$$

Together with (3.3) it yields

$$i_1^E(\psi)=i_1^E(\psi_{-s}) \tag{3.4}$$

for sufficiently small $s \in (0, 1]$, and then for all $s \in (0, 1]$ by (3.2) and Theorem 5.7. Thus from Theorem 5.7, (3.2), and (3.4), for $s \in (0, 1]$ we obtain

$$i_1(z) = i_1(\psi) = i_1(\psi_{-s}) = i_1^E(\psi_{-s}) + n = i_1^E(\psi) + n = i_1^E(u) + n.$$

This proves the lemma. ▮

Remark 3.1. Lemma 1.3 holds also for any periodic time dependent convex Hamiltonians by the same proof.

Proof of Lemma 1.4. Define

$$\gamma(t) = \psi(h^{(2-\alpha)/\alpha} t), \qquad \forall t \in [0, +\infty), \tag{3.5}$$

where ψ is defined at the beginning of the proof of Case 2 of Lemma 1.3. Then using the positive homogeneity of H''_α, we obtain that γ: $[0, +\infty) \to \mathrm{Sp}(2n)$ is the fundamental solution of the system (1.2) with $B(t) = H''_\alpha(x(t))$, i.e., γ coincides with the associated symplectic path γ_x of x. Thus by (1.8) and (3.5) there holds $\gamma_x(m\tau) = \psi(1)$. This implies $\nu_{m\tau}(x^m) = \nu_1(z)$. Since $\gamma_x|_{[0, m\tau]}$ is only a rescaling of $\psi|_{[0, 1]}$, they are geometrically the same path in $\mathrm{Sp}(2n)$. This yields $i_{m\tau}(x^m) = i_{m\tau}(\gamma_x|_{[0, m\tau]}) = i_1(\psi|_{[0, 1]}) = i_1(z)$, and completes the proof. ▮

Next we study the structure of the associated symplectic matrix $\gamma_x(\tau)$ of (τ, x) for any $(\tau, x) \in \mathscr{J}(\Sigma, \alpha)$ with $1 < \alpha < 2$.

Now by Lemma I.7.3 of [Ek3], the non-zero vectors $\dot{x}(0)$ and $x(0)$ satisfy

$$\begin{cases} \gamma_x(\tau)\, \dot{x}(0) = \dot{x}(0), \\ \gamma_x(\tau)\, x(0) = \tau(\alpha - 2)\, \dot{x}(0) + x(0). \end{cases} \tag{3.6}$$

Define

$$\xi_1 = \tau(\alpha - 2)\, \dot{x}(0), \qquad \xi_2 = x(0). \tag{3.7}$$

Then (3.6) becomes

$$\begin{cases} \gamma_x(\tau)\, \xi_1 = \xi_1, \\ \gamma_x(\tau)\, \xi_2 = \xi_1 + \xi_2, \end{cases} \tag{3.8}$$

LEMMA 3.2. *For* $1 < \alpha < 2$ *and* $(\tau, x) \in \mathscr{J}(\Sigma, \alpha)$, *there exist matrices* $P \in \mathrm{Sp}(2n)$ *and* $M \in \mathrm{Sp}(2n-2)$ *such that there holds*

$$\gamma_x(\tau) = P(N_2(1) \diamond M)\, P^{-1}, \tag{3.9}$$

where $N_2(1)$ *is defined by* (2.3). *Note that there holds* $N_2(1) \in \mathrm{Sp}(2)^0_-$ (*cf. Section* 5).

Proof. Fix $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$ and defien ξ_1 and ξ_2 by (3.7). We carry out the proof in three steps.

Step 1. Since $x = x(t)$ is a solution of (1.4), we have $x(0) \in \Sigma$ and $\dot{x}(0) = JH'_\alpha(x(0))$. Since Σ is convex and C^2, by a direct computation and the fact $1 < \alpha < 2$ we obtain

$$\xi_1^* J \xi_2 = \tau(\alpha - 2)\langle H'_\alpha(x(0)), x(0)\rangle < 0. \tag{3.10}$$

Step 2. Now suppose $\{\xi_1, \xi_2, ..., \xi_p\}$ form a Jordan block of $\gamma_x(\tau)$ belonging to the eigenvalue 1, i.e., setting $\xi_0 = 0$, there hold

$$\gamma_x(\tau)\, \xi_i = \xi_i + \xi_{i-1}, \qquad \forall 1 \leqslant i \leqslant p. \tag{3.11}$$

For $1 \leqslant i, j \leqslant p$ by (3.11) we have

$$\xi_i^* J \xi_j = (M\xi_i)^* J(M\xi_j) = \xi_i^* J \xi_j + \xi_{i-1}^* J \xi_j + \xi_i^* J \xi_{j-1} + \xi_{i-1}^* J \xi_{j-1}.$$

This yield

$$\xi_{i-1}^* J \xi_{j-1} + \xi_{i-1}^* J \xi_j + \xi_i^* J \xi_{j-1} = 0, \qquad \forall 1 \leqslant i, \quad j \leqslant p, \tag{3.12}$$

$$\xi_i^* J \xi_j = 0, \qquad \forall 1 \leqslant i \leqslant p - j, \quad 1 \leqslant j \leqslant \left[\frac{p}{2}\right], \tag{3.13}$$

where $[a] = \max\{m \in \mathbf{Z} \mid m \leqslant a\}$ for any $a \in \mathbf{R}$, and (3.13) follows from (3.12) by induction.

Thus from (3.10) and (3.13) we must have $p = 2$, i.e., ξ_1 and ξ_2 form a Jordan block of $\gamma_x(\tau)$ belonging to the eigenvalue 1.

Step 3. Define

$$\delta_1 = \frac{1}{\sqrt{|\xi_1^* J \xi_2|}}\, \xi_1 \qquad \text{and} \qquad \delta_2 = \frac{1}{\sqrt{|\xi_1^* J \xi_2|}}\, \xi_2. \tag{3.14}$$

Then there hold

$$\delta_1^* J\, \delta_2 = -1, \qquad F \equiv \operatorname{span}\{\delta_1, \delta_2\} = \operatorname{span}\{\xi_1, \xi_2\}, \tag{3.15}$$

i.e., $\{\delta_1, \delta_2\}$ form a symplectic base for F. Denote by K the $2n \times 2$ matrix formed by δ_1 and δ_2 as the first and the second columns. From (3.8), (3.14), and (3.15), we obtain $\gamma_x(\tau)\, K = KN_2(1)$. Now we can extend K to a matrix $P \in \mathrm{Sp}(2n)$ such that δ_1 and δ_2 form the first and the $(n+1)$-st columns of P and for some $M \in \mathrm{Sp}(2n-2)$ such that (3.9) holds.

By direct computation we obain $N_2(1) \in \mathrm{Sp}(2)^0_-$. This proves the lemma. ∎

Remark 3.3. Lemma 3.2 holds for $\alpha=2$ or $\alpha>2$ with $N_2(1)$ in (3.9) being replaced by I_2 or $N_2(-1)$ respectively.

Note that Lemma I.7.3 of [Ek3] and the discussions on pages 407–408 of [Ek2] only proves that the vectors $\dot{x}(0)$ and $x(0)$ satisfy (3.6). Our Lemma 3.2 further proves that they actually form a 2-dimensional invariant subspace of $\gamma_x(\tau)$ belonging to the eigenvalue 1, whether x is hyperbolic or not. Specifically our Lemma 3.2 on $\gamma_x(\tau)$ is stronger than (7) of [Ek2].

4. HYPERBOLIC AND NONHYPERBOLIC CLOSED CHARACTERISTICS

Fix a hypersurface $\Sigma\in\mathscr{H}(2n)$ and $1<\alpha<2$.

PROPOSITION 4.1. *Suppose $(\tau, x)\in\mathscr{J}(\Sigma,\alpha)$ is hyperbolic. Denote by γ_x its associated symplectic path. Then*

(1) *there hold*

$$n\leqslant i_\tau(x),\quad i_{m\tau}(x^m)=m(i_\tau(x)+1)-1,\quad \nu_{m\tau}(x^m)=1,\qquad \forall m\in\mathbf{N}. \tag{4.1}$$

(2) *$i_\tau(x)$ is even if* (2.16) *holds for $\gamma_x(\tau)$, and $i_\tau(x)$ is odd if* (2.19) *holds for $\gamma_x(\tau)$.*

Proof. (1) and (2) follow from (1.13), subcases 1.1 and 1.2 of Theorem 2.1 and Lemma 3.2. ∎

Remark 4.2. Similarly by Remark 3.3, if we choose $\alpha=2$, the subcases 2.1 and 2.2 of Theorem 2.1 happen. If we choose $\alpha>2$, the subcases 3.1 and 3.2 of Theorem 2.1 happen. Therefore for hyperbolic $(\tau, x)\in\mathscr{J}(\Sigma,\alpha)$, there hold $i_\tau(x)\geqslant n$ and

$$\begin{aligned}
&i_{m\tau}(x^m)=m(i_\tau(x)+1)-1, && \nu_{m\tau}(x^m)=2,\quad \forall m\in\mathbf{N}, && \text{if}\quad \alpha=2,\\
&i_{m\tau}(x^m)=mi_\tau(x), && \nu_{m\tau}(x^m)=1,\quad \forall m\in\mathbf{N}, && \text{if}\quad \alpha>2.
\end{aligned}$$

Remark 4.3. The iteraion formula of Ekeland index theory for hyperbolic closed characterestics on $\Sigma\in\mathscr{H}(2n)$ with $1<\alpha<2$ was given by Corollary IV.8 of [Ek1] as $i_k^E=k(i_1^E+n)-n$, and by (IV.17) of [Ek2] as $i_k^E=i_1^E+(k-1)n$. Both of these two are incorrect, and the proof of Theorem IV.1 in [Ek2] is not complete. By our (4.1) and Lemmata 1.3 and 1.4 when $\Sigma\in\mathscr{H}(2n)$ and $1<\alpha<2$ this formula should be corrected to $i_k^E=k(i_1^E+n+1)-n-1$ for all $k\in\mathbf{N}$.

PROPOSITION 4.4. *For $n \geq 2$ and $1 < \alpha < 2$, under the condition of Theorem 1.9, there exists an integer $q_0 \geq n$ depending only on Σ and α such that there hold*

$$\mathscr{I}_h(\Sigma, \alpha) \subset \mathscr{B}(\Sigma, \alpha), \tag{4.2}$$

where we define

$$\mathscr{B}(\Sigma, \alpha) = \left\{ m(2q+1) - 1 \,\middle|\, m \in \mathbf{N}, \frac{n}{2} \leqslant q \leqslant q_0 \right\}, \qquad \text{if} \quad n \in 2\mathbf{N}, \tag{4.3}$$

$$\mathscr{B}(\Sigma, \alpha) = \left\{ m(2q) - 1, 2m(2q+1) - 1 \,\middle|\, m \in \mathbf{N}, \frac{n+1}{2} \leqslant q \leqslant q_0 \right\}, \qquad \text{if} \quad n \in 2\mathbf{N} + 1. \tag{4.4}$$

Proof. Denote by b_k's the constants depending only on Σ and α in this proof. By the condition of Theorem 1.9, there exists a constant b_1 such that for any hyperbolic $(\tau, x) \in \mathscr{V}(\Sigma, \alpha)$ there holds $0 < \tau \leqslant b_1$. Thus there holds

$$A(\tau, x) \equiv \frac{1}{2} \int_0^\tau \langle \dot{x}, Jx \rangle \, dt = \frac{\alpha}{2} \tau \leqslant \frac{\alpha}{2} b_1. \tag{4.5}$$

For hyperbolic $(\tau, x) \in \mathscr{V}(\Sigma, \alpha)$, by Definition 1.5, Remark 1.6, and Lemma V.3.12 of [Ek3] there exists positive constants b_2, b_3, and some $k \in \mathbf{N}$ depending on (τ, x) such that there holds

$$\left| \frac{\hat{\imath}(x)}{b_2 A(\tau, x)} - k \, |c_k|^{(2-\alpha)/\alpha} \right| \leqslant b_3 \, |c_k|^{(2-\alpha)/(2\alpha)}, \tag{4.6}$$

where $\hat{\imath}(x)$ is the mean Maslov-type index of (τ, x) defined by $\hat{\imath}(x) = \lim_{m \to \infty} i_{m\tau}(x^m)/m$. This limit exists by [Lo6] or Lemmata 1.3, 1.4, and Chapter I of [Ek3]. By the definition of c_k's, the right hand side of (4.6) is bounded by a constant b_4. By Lemma V.3.9 of [Ek3] the term $k \, |c_k|^{(2-\alpha)/\alpha}$ in (4.6) is bounded by a constant b_5. Together with (4.5) this yields a constant b_6 depending only on Σ and α such that there holds

$$\hat{\imath}(x) \leqslant b_6. \tag{4.7}$$

Since (τ, x) is hyperbolic, by (4.1) and (1.13) we obtain

$$2 \leqslant n \leqslant i_\tau(x) = \hat{\imath}(x) - 1 \leqslant b_6 - 1. \tag{4.8}$$

By (4.1) and Definition 1.5, there must hold

$$2k - 2 + n = i_{m\tau}(x^m), \qquad \forall \text{ hyperbolic}(\tau, x) \in \mathscr{V}_{m,k}(\Sigma, \alpha). \tag{4.9}$$

If n is even, so is $2k-2+n$. By (4.1) and (4.9) all hyperbolic $(\tau, x) \in \mathscr{J}(\Sigma, \alpha)$ with odd $i_\tau(x)$ can not be variationally visible. Therefore by (4.1), (4.8), and (4.9), there exists $q_0 \in \mathbf{N}$ such that any hyperbolic $(\tau, x) \in \mathscr{V}(\Sigma, \alpha)$ must satisfy $i_\tau(x) = 2q$ with $q \in [n/2, q_0]$ and $i_{m\tau}(x^m) = m(2q+1) - 1$ for all $m \in \mathbf{N}$. Thus (4.2) and (4.3) follow from the evenness of $2k-2+n$ and (4.9).

If n is odd, so is $2k-2+n$. By (4.1), (4.8), and (4.9), there exists $q_0 \in \mathbf{N}$ such that for any hyperbolic $(\tau, x) \in \mathscr{V}(\Sigma, \alpha)$, if $i_\tau(x)$ is odd, there must hold $i_\tau(x) = 2q-1$ with $q \in [2, q_0]$ and $i_{m\tau}(x^m) = m(2q) - 1$ for all $m \in \mathbf{N}$. If $i_\tau(x)$ is even, there must hold $i_\tau(x) = 2q$ with $q \in [(n+1)/2, q_0]$ and $i_{m\tau}(x^m) = 2m(2q+1) - 1$ for all $m \in \mathbf{N}$. Thus (4.2) and (4.4) hold. ∎

Now we can give

Proof of Theorem 1.9. We prove (1.24) indirectly by assuming

$$\mathscr{I}(\Sigma, \alpha) = \mathscr{I}_h(\Sigma, \alpha). \tag{4.10}$$

Define $Q(n) = 2\mathbf{N} - 2 + n$. We continue our study in two cases.

Case 1. n is even.

In this case $Q(n) \subset 2\mathbf{N}$. By the infinitude of prime integers, we can choose a prime integer $p > \max\{n, 2q_0+1\}$ with q_0 defined in (4.2), and define $k = (p+1-n)/2$. By (1.23), (4.2), (4.3), and (4.10), we obtain $Q(n) \subset \mathscr{B}(\Sigma, \alpha)$. Thus there exist integers $q \in [1, q_0]$ and $m \in \mathbf{N}$ such that there holds $p = 2k-1+n = m(2q+1)$. This contradicts to the choice of p.

Case 2. $n \geqslant 2$ is odd.

In this case $Q(n) \subset 2\mathbf{N} - 1$. Choose a prime integer $p > \max\{n, 2q_0+1\}$ with q_0 defined in (4.4), and define $k = (2p+1-n)/2$. Then by (1.23), (4.2), (4.4), and (4.10), we obtain $Q(n) \subset \mathscr{B}(\Sigma, \alpha)$. Thus there holds either $p = \frac{1}{2}(2k-1+n) = mq$ for some integers $m \in \mathbf{N}$ and $q \in [2, q_0]$, or $p = \frac{1}{2}(2k-1+n) = m(2q+1)$ for some integers $m \in \mathbf{N}$ and $q \in [1, q_0]$. Both of the two equalities contradict to the choice of p.

The proof is complete. ∎

5. APPENDIX: THE MASLOV-TYPE INDEX THEORY FOR SYMPLECTIC PATHS

The Maslov-type index theory for nondegenerate continuous paths starting from the identity matrix I in $\mathrm{Sp}(2n)$ was established by C. Conley and E. Zehnder in [CZ] for $n \geqslant 2$, and by E. Zehnder and the author in [LZ] for $n = 1$. For degenerate paths which are foundamental solutions of the linear system (1.2) with $B(t)$ being symmetric continuous and τ-periodic $2n\times 2n$ real matrix function, this index theory was established by the author in [Lo1] and C. Viterbo in [Vi2] independently, and then in [Lo5–7] it was extended to all continuous degenerate paths. In this section we give a brief introduction of this index theory without proofs. For details we refer to [CZ], [LZ], and [Lo1–7].

As usual the symplectic group $\mathrm{Sp}(2n)$ consists of all $2n\times 2n$ real matrices M satisfying $M^TJM=J$, where M^T is the transpose of M. Define

$$\mathrm{Sp}(2n)^{\pm}=\{M\in\mathrm{Sp}(2n)\mid \pm\det(M-I_{2n})<0\},$$

$$\mathrm{Sp}(2n)^{*}=\mathrm{Sp}(2n)^{+}\cup\mathrm{Sp}(2n)^{-},\qquad \mathrm{Sp}(2n)^{0}=\mathrm{Sp}(2n)\backslash\mathrm{Sp}(2n)^{*},$$

$$\mathrm{Sp}(2)^{0}_{\pm}=\{M\in\mathrm{Sp}(2)^{0}\mid \sigma(MR(\theta))\subset\mathbf{U}\backslash\mathbf{R}\text{ for }0<\pm\theta<\pi/2\},$$

where $R(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ for $\theta\in\mathbf{R}$. For $m\in\mathbf{N}$ we define

$$\mathrm{Sp}(2n)^{*}_{m}=\{M\in\mathrm{Sp}(2n)\mid \det(M^m-I_{2n})\neq 0\},$$

$$\mathrm{Sp}(2n)^{0}_{m}=\mathrm{Sp}(2n)\backslash\mathrm{Sp}(2n)^{*}_{m}.$$

Note that $\mathrm{Sp}(2n)^{*}_{m}$ is an open subset of $\mathrm{Sp}(2n)$ in the topology induced from $\mathbf{R}^{4n^2}$.

Fix $\tau>0$. Define

$$\mathscr{P}_\tau(2n)=\{\gamma\in C([0,\tau],\mathrm{Sp}(2n))\mid \gamma(0)=I_{2n}\},$$

$$\mathscr{P}^{*}_\tau(2n)=\{\gamma\in\mathscr{P}_\tau(2n)\mid \gamma(\tau)\in\mathrm{Sp}(2n)^{*}\},\qquad \mathscr{P}^{0}_\tau(2n)=\mathscr{P}_\tau(2n)\backslash\mathscr{P}^{*}_\tau(2n),$$

$$P_\tau(2n)=\{\gamma\in\mathscr{P}_\tau(2n)\cap C^1([0,\tau],\mathrm{Sp}(2n))\mid \dot\gamma(\tau)=\dot\gamma(0)\,\gamma(\tau)\}.$$

Note that $P_\tau(2n)$ contains the fundamental solutions of all linear Hamiltonian systems (1.2) with $B(t)$ being continuous, symmetric, and τ-periodic.

For any two paths ψ and $\phi\colon[0,1]\to\mathrm{Sp}(2n)$ with $\psi(1)=\phi(0)$ we define their joint path by

$$\phi*\psi(t)=\begin{cases}\psi(2t), & \text{if}\quad 0\leqslant t\leqslant 1/2,\\ \phi(2t-1) & \text{if}\quad 0\leqslant t\leqslant 1/2.\end{cases}$$

Given any two even order matrices of square block form:

$$M_1=\begin{pmatrix}A_1 & B_1\\ C_1 & D_1\end{pmatrix}_{2i\times 2i},\qquad M_2=\begin{pmatrix}A_2 & B_2\\ C_2 & D_2\end{pmatrix}_{2j\times 2j},$$

we define the $\diamond$-product of M_1 and M_2 to be the $2(i+j)\times 2(i+j)$ matrix $M_1 \diamond M_2$ given by

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$

We denote by $M_1^{\diamond n}$ the n-fold $\diamond$-product of M_1. Note that the $\diamond$-product of two symplectic matrices is still symplectic.

Definition 5.1 (cf. [Lo1]). For every $\gamma \in \mathscr{P}_\tau(2n)$, we define $\nu_\tau(\gamma) = \dim\ker(\gamma(\tau) - I_{2n})$

Definition 5.2 (cf. [Lo1]). Given two paths γ_0 and $\gamma_1 \in \mathscr{P}_\tau(2n)$, if there is a map $\delta \in C([0,1]\times[0,\tau], \mathrm{Sp}(2n))$ such that $\delta(0,\cdot) = \gamma_0(\cdot)$, $\delta(1,\cdot) = \gamma_1(\cdot)$, $\delta(s,0) = I_{2n}$, and $\nu_\tau(\delta(s,\cdot))$ is constant for $0 \leqslant s \leqslant 1$, then γ_0 and γ_1 are *homotopic on* $[0,\tau]$ *along* $\delta(\cdot,\tau)$ and we write $\gamma_0 \sim \gamma_1$ on $[0,\tau]$ along $\delta(\cdot,\tau)$. This homotopy possesses fixed end points if $\delta(s,\tau) = \gamma_0(\tau)$ for all $s \in [0,1]$.

Definition 5.3. Given M_0 and $M_1 \in \mathrm{Sp}(2n)$, if there is a path $h \in C([0,1], \mathrm{Sp}(2n))$ such that $h(0) = M_0$, $h(1) = M_1$, and $\dim\ker(h(t) - I_{2n}) = constant$, Then M_0 and M_1 are *homotopic along* h and we write $M_0 \simeq M_1$.

As is well known, every $M \in \mathrm{Sp}(2n)$ has its unique polar decomposition $M = AU$, where $A = (MM^T)^{1/2}$ is symmetric positive definite and symplectic, U is orthogonal and symplectic. Therefore U has the form $U = \left(\begin{smallmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{smallmatrix}\right)$, where $u = u_1 + \sqrt{-1}\, u_2 \in \mathscr{L}(\mathbf{C}^n)$ is a unitary matrix. So for every path γ in $\mathscr{P}_\tau(2n)$ we can associate a path $u(t)$ in the unitary group on $\mathbf{C}^n$ to it. If $\Delta(t)$ is any continuous real function satisfying $\det u(t) = \exp(\sqrt{-1}\,\Delta(t))$, the difference $\Delta(\tau) - \Delta(0)$ depends only on γ but not on the choice of the function $\Delta(t)$. Therefore we may define the mean rotation number of γ on $[0,\tau]$ by $\Delta_\tau(\gamma) = \Delta(\tau) - \Delta(0)$.

Lemma 5.4 (cf. [Lo1, 5]). *If γ_0 and $\gamma_1 \in \mathscr{P}_\tau(2n)$ possess common end point $\gamma_0(\tau) = \gamma_1(\tau)$, then $\Delta_\tau(\gamma_0) = \Delta_\tau(\gamma_1)$ if and only if $\gamma_0 \sim \gamma_1$ on $[0,\tau]$ with fixed end points.*

Let $D(a) = \mathrm{diag}\{a, a^{-1}\}$ for $a \in \mathbf{R}\backslash\{0\}$. For any $\gamma \in \mathscr{P}_\tau^*(2n)$, we can connect $\gamma(\tau)$ to $D(2)^{\diamond n}$ or $D(-2) \diamond D(2)^{\diamond(n-1)}$ by a path β in $\mathrm{Sp}(2n)^*$ to get a joint path $\beta * \gamma$. Then $k \equiv \Delta_\tau(\beta * \gamma)/\pi \in \mathbf{Z}$ and is independent of the choice of the path β. In this case we write $\gamma \in \mathscr{P}^*_{\tau,k}(2n)$. These $\mathscr{P}^*_{\tau,k}(2n)$'s give a homotopy classification of $\mathscr{P}^*_\tau(2n)$.

DEFINITION 5.5 (cf. [CZ], [LZ]). If $\gamma \in \mathscr{P}^*_{\tau,k}(2n)$, we define $i_\tau(\gamma) = k$.

DEFINITION 5.6 (cf. [Lo1, 5, 6, 7]). For $\gamma \in \mathscr{P}^0_\tau(2n)$ define

$$i_\tau(\gamma) = \inf\{i_\tau(\beta) \mid \beta \in \mathscr{P}^*_\tau(2n) \text{ and } \beta \text{ is sufficiently close to } \gamma \text{ in } \mathscr{P}_\tau(2n)\},$$

where the topology of $\mathscr{P}_\tau(2n)$ is the C^0-topology induced from the topology of $\mathrm{Sp}(2n)$.

THEOREM 5.7 (cf. [Lo1, 5, 6]). *For $\gamma \in \mathscr{P}^0_\tau(2n)$ define*

$$i^+_\tau(\gamma) = \sup\{i_\tau(\beta) \mid \beta \in \mathscr{P}^*_\tau(2n) \textit{ and } \beta \textit{ is sufficiently close to } \gamma \textit{ in } \mathscr{P}_\tau(2n)\}.$$

Then for every $\beta \in \mathscr{P}^*_\tau(2n)$ which is sufficiently close to γ, there holds

$$-\infty < i_\tau(\gamma) \leqslant i_\tau(\beta) \leqslant i^+_\tau(\gamma) = i_\tau(\gamma) + \nu_\tau(\gamma) < +\infty. \tag{5.1}$$

Specially, there exists a one-parameter continuous family $\gamma_s \in \mathscr{P}_\tau(2n)$ of paths with $s \in [-1, 1]$ satisfying γ_s converges to $\gamma_0 = \gamma$ in C^0 as $s \to 0$, $\gamma_s \in \mathscr{P}^*_\tau(2n)$ for $s \in [-1, 1] \backslash \{0\}$, and there hold

$$i_\tau(\gamma_{-s}) = i_\tau(\gamma) = i_\tau(\gamma_s) - \nu_\tau(\gamma), \qquad \forall s \in (0, 1]. \tag{5.2}$$

Moreover, if $\gamma \in \mathscr{P}^0_\tau(2n) \cap P_\tau(2n)$, we can further require $\gamma_s \in P_\tau(2n)$ for all $s \in [-1, 1]$ and γ_s converging to γ in C^1 as $s \to 0$.

DEFINITION 5.8 (cf. [Lo1, 5, 7]). For every path $\gamma \in \mathscr{P}_\tau(2n)$, Definitions 5.1, 5.5, and 5.6 assign a pair of integers $(i_\tau(\gamma), \nu_\tau(\gamma)) \in \mathbf{Z} \times \{0, ..., 2n\}$ to it. This pair of integers is called the *Maslov-type index* of γ.

PROPOSITION 5.9 (cf. [Lo5, 7]). *For any two paths γ_0 and $\gamma_1 \in \mathscr{P}_\tau(2n)$ with $i_\tau(\gamma_0) = i_\tau(\gamma_1)$, suppose that there exists a continuous path $h\colon [0, 1] \to \mathrm{Sp}(2n)$ such that $h(0) = \gamma_0(\tau)$, $h(1) = \gamma_1(\tau)$, and $\dim\ker(h(s) - I_{2n}) = \nu_\tau(\gamma_0)$ for all $s \in [0, 1]$. Then $\gamma_0 \sim \gamma_1$ on $[0, \tau]$ along h.*

PROPOSITION 5.10 (cf. [Lo5, 7]). *For any paths γ_0 and $\gamma_1 \in \mathscr{P}_\tau(2n)$, if $\gamma_0 \sim \gamma_1$ on $[0, \tau]$, then*

$$\nu_\tau(\gamma_0) = \nu_\tau(\gamma_1) \qquad \textit{and} \qquad i_\tau(\gamma_0) = i_\tau(\gamma_1).$$

PROPOSITION 5.11 (cf. [Lo7]). *For any paths $\gamma_0 \in \mathscr{P}_\tau(2n_0)$ and $\gamma_1 \in \mathscr{P}_\tau(2n_1)$, there hold $\gamma_0 \diamond \gamma_1 \in \mathscr{P}_\tau(2n_0 + 2n_1)$ and*

$$i_\tau(\gamma_0 \diamond \gamma_1) = i_\tau(\gamma_0) + i_\tau(\gamma_1).$$

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